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Atomic Varieties of Sets with Relative Inverses

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The generalization of the construction of the lattice of varieties for partial algebras is used for sets with relative inverses. There are many quantum structures representable by sets with relative inverses (orthomodular lattices, orthoalgebras, D-posets, test spaces, . . .). Varieties covering the trivial variety are investigated for the case of closed (strongest type) subalgebras and closed homomorphisms. Some similar results for weaker types are given. The context with set representation problems is considered for the set-theoretic difference operations.

1. DEFINITIONS AND BASIC RESULTS

The notion of the set with relative inverses was introduced by Kalmbach and Riečanová (1994). The following definition is the list of axioms for the basic notion of this paper.

Definition (We will use the abbreviation "RI-sets"). An RI-set is a partial algebra of similarity type

$$(2, 0)$$

$$(X, \boxminus, O)$$

with

X . . . . . . nonempty carrier set
$\boxminus$ . . . . . . partial binary operation relative inverse
O . . . . . . special element (generalization of the empty set)

with the following rules (axioms):

(I) $a \boxminus O = a$.
(II) $a \boxminus a = O$.

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(III) If $b \boxminus a$ is evaluable, then $b \boxminus (b \boxminus a)$ is evaluable.
(IV) If $b \boxminus a$ and $a \boxminus c$ are evaluable, then $b \boxminus c$ is evaluable.
(V) $(a \boxminus b) \boxminus c = (a \boxminus c) \boxminus b$.

The last one is the existence equation (if all terms of one side are evaluable, then both sides are evaluable and the equation holds).

The calculus for creating varieties for partial algebras is developed in Burmeister (1986). By using this tool we obtain some results for a unified point of view for frequent quantum structures. In Burmeister (1986) three degrees of homomorphisms and subalgebras are distinguished. We begin with the notions of the closed homomorphism and the closed subalgebra.

Definition of the weak homomorphism: Let A, B be RI-sets and ϕ: $A \to B$ be a mapping into B. Then ϕ is a weak homomorphism if $\phi(b \boxminus_A a) = \phi(b) \boxminus_B (a)$ and ϕ preserves O.

Definition of the closed homomorphism: A closed homomorphism is a homomorphism with: if $[\phi(b) \boxminus_B \phi(a)]$ is evaluable, then for every $u, v \in A$ such that $\phi(b) = \phi(u)$ and $\phi(a) = \phi(v)$, $[u \boxminus_A v]$ must be evaluable.

Definition of the closed subalgebra: B is a closed subalgebra of A if the canonical embedding i: $B \to A$ ...... $i(x) = x$ is an injective closed homomorphism.

It is easy to see that there are three possibilities of two-element carrier sets: (R_1: $0 \boxminus 1 = 0$), (R_2: $0 \boxminus 1 = 1$), (R_3: $0 \boxminus 1$ is not evaluable). The property "there is a nonevaluable couple with first member 0" is preserved under the direct product construction and homomorphism image. This property is preserved under the subalgebra operator in the following sense: every algebra with this property has a two-element subalgebra with this property. This implies that the variety generated by R_3 is atomic (it covers the trivial variety). R_2 is a total algebra and satisfies the equation $0 \boxminus x = x$. This implies that all members of the variety generated by R_2 satisfy this equation. The variety generated by R_2 is the variety with known equational characterization by $0 \boxminus x = x$. This fact is the corollary of the basic theory for varieties of Abelian groups. This variety is evidently atomic. The atomicity of the variety generated by R_1 is proved by fact that every RI-set satisfying $0 \boxminus x = 0$ has a two-element subalgebra of this type.

If the set of all nonevaluable couples of an RI-set is nonempty, then this RI-set is contained only in varieties containing R_3. The other atoms in the lattice of varieties of RI-sets are generated by total algebras and all of its members are total algebras. Kalmbach and Riečanova (n.d.) investigate the notion of Abelian RI-sets characterized by a generalized equation formulated by using the implication $x \boxminus a = y \boxminus a$ implies that $x = y$. The

Abelian RI-set is the generalization of the Abelian group and the equational characterization of atoms in the lattice of varieties of Abelian groups can be translated into the language of RI-sets and the system of the atomic varieties of Abelian groups can be embedded into the system of atoms of varieties of RI-sets. The variety generated by R_2 is of this type.

The problem of the equational characterization of the variety generated by R_1 is a nice example for the calculus of varieties of RI-sets. The variety generated by R_1 is a subvariety of the one characterized by the equation $0 \boxminus x = 0$. It is the proper subvariety. The following example is on the three-element carrier set $\{0, 1, 2\}$ with the list of results of operations (the couples with results determined by axioms and characterizing equation are omitted): $1 \boxminus 2 = 0$, $2 \boxminus 1 = 1$ is the element proving the difference between the equational characterization and the description by the generating set. There are two equations satisfied in the variety generated by R_1 such that the problem of equational characterization is nearly solved:

$$x \boxminus (x \boxminus y) = y \boxminus (y \boxminus x), \qquad x \boxminus (y \boxminus x) = x$$

The variety of RI-sets determined by these two equations is the variety whose members can be characterized as "the RI-sets with order relation ($x \leq y$ iff $x \boxminus y = 0$) and every interval of type $[0, z]$ is a Boolean algebra." Stone's representation theorem for Boolean algebras enables the relative modification of set representations on subintervals. Two methods for generalization on RI-sets are sufficiently natural: One is a generalization of the notion of maximal ideal and is similar to Stone's theorem. The second is to obtain the representation on every interval and make compatible all these representations. The first way can be used without technical difficulties for RI-sets with the property "for every element x there is an atom c such that $c \leq x$ holds." There are technical complications in proving an isomorphism for the canonical mapping from an RI-set to the set of all maximal ideals for RI-sets such that maximal ideals cannot be described by atoms. The second method is without restrictions, but the problem is to design factorizations for special types of elements. More precisely, a poset from a variety generated by R_1 is a semilattice with defined meet and with a weak form of the definition of the join: "if there is an upper bound for a two-element set $\{x, y\}$, then there is a join of these elements." If an element is a meet of the couple of elements without the join and it is the coatom, then this element may determine the maximal ideal which is the member of the one-element representing set for two or more atoms in the RI-set.

The problem of equational characterization of the variety generated by R_1 is open and the following formulation is precise:

Let **V** be a variety of RI-sets characterized by the equations

$$x \boxminus (x \boxminus y) = y \boxminus (y \boxminus x)$$

$$x \boxminus (y \boxminus x) = x$$

We define the following notions: the order relation on every element of **V** by $x \leq y$ iff $x \boxminus y = 0$; the meet in this poset for every couple $\{x, y\}$ by $x \wedge y = x \boxminus (x \boxminus y)$; the join in this poset for the couple $\{x, y\}$ with common upper bound u by $x \vee y = u \boxminus [(u \boxminus x) \wedge (u \boxminus y)]$. (This definition is correct—it does not depend on the choice of upper bound.) It is known that (1) every subinterval $[0, s]$ is an ortholattice with the orthocomplementation $x' = s \boxminus x$, (2) this ortholattice is orthomodular, (3) every couple of elements of this interval is compatible ($[0, s]$ is a Boolean algebra).

The problem is to prove that every element of **V** is set-representable such that the set representation is considered with the operation

$$A - B = \{x \in A; x \notin B\}$$

(B not need be a subset of A) and the empty set as the zero element.

The set representation with the structure of the variety generated by R_1 is based on the fact that the direct product of arbitrary many copies of R_1 is an RI-set with a system of subsets of some set as a carrier set, the operator of making the subalgebra preserves this property, and the homomorphic image is equivalent to the isomorphic copy. All elements of **V** are set representable. The opposite case that every set-representable algebra is the element of **V** is obvious.

If some RI-set U generates the variety such that for every variety from the list of atomic varieties considered above this variety is not contained in the variety by generated U, then U must to satisfy the following conditions: (a) U is a total algebra, (b) $0 \boxminus x$ is not an element of $\{0, x\}$, (c) there is no subalgebra of U such that it is an Abelian group in sense described in (Kalmbach and Riečanová, n.d.) (it is a natural transformation of the operation $\boxminus$ to the group operation: $x \circ y = z$ iff $z \boxminus x = y$).

We will consider the structure of RI-sets with this property by introducing a special mapping $\phi\colon U \to U$ defined by the equation $\phi(x) = 0 \boxminus x$. For every RI-set the following equation holds:

$$0 \boxminus [0 \boxminus (0 \boxminus a)] = 0 \boxminus a$$

To prove this identity we use admissible transformations starting from the evident fact $(0 \boxminus a) \boxminus (0 \boxminus a) = 0$. The precise procedure of computation will be omitted. If we use this equation for the analysis of orbits of the mapping ϕ, we obtain the result that by starting from the arbitrary element and making one step we arrive at the element of a two-element orbit. If the mapping ϕ is injective, then the fact that we consider total algebras implies the sufficiency of the Abelian criterion ($c \boxminus a = d \boxminus a$ implies $c = d$); see Kalmbach and Riečanová (1994) for the result that the considered RI-set is a Boolean algebra. The injectivity of ϕ implies that this criterion holds. Atomic varieties for Boolean algebras are considered above. In the case the mapping ϕ is not injective, there are many configurations, and results of the analysis of orbits of the mapping ϕ and the classification of varieties are unknown to the author.

2. WEAK TYPES OF SUBALGEBRA AND HOMOMORPHISM

In the case of the weak homomorphism and the subalgebra induced by the identical embedding to the weak homomorphism there are only two varieties: the trivial one and the variety of all nontrivial RI-sets. This case is not interesting.

In the case of the weak homomorphism and a subalgebra of arbitrary degree in the sense of the classification in Burmeister (1986) the RI-set R_3 generates the variety containing all nontrivial RI-sets. This case is not interesting.

Let us introduce the notion of full homomorphism (Burmeister, 1986):

A full homomorphism is a homomorphism with: $[\phi(b) \boxminus_B \phi(a)]$ is evaluable with result in $\phi(A)$ only in the case that $\phi(b) = \phi(u)$ and $\phi(a) = \phi(v)$ for some evaluable $[u \boxminus_A v]$.

If we consider the case of full homomorphism and closed subalgebra then atomic varieties of Abelian groups (in the sense introduced above) are atomic but not different because they are equivalent to the variety generated by R_3.

The case of full homomorphism is very interesting, but the classification is not finished yet.

3. SET REPRESENTATION ASPECTS

The set representation is based on the fact that to be an element of some set is a statement with two Boolean values. This implies that set representations by characteristic functions are representations as subalgebras of the direct product of two-element RI-sets. There are three possibilities for two-element RI-sets induced in three ways for the definition of the partial binary operation on set-representable RI-sets. These are the difference on the arbitrary couple of sets (derived from R_1), the symmetric difference (derived from R_2), and the difference between some set and its proper subset (derived from R_3). The R_1 case was investigated in the first part of this article, the R_2 case is the domain of classical measure theory, and the R_3 case is the basis for the generalized orthomodular poset representation theory. Generalization of the orthomodular poset is in the sense that every interval of a generalized orthomodular poset is the orthomodular poset but the common upper bound is not necessary. The criterion by the order-determining system of states (Gudder, 1979) is an example of the application of variety calculus in quantum logic theory.

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